

# Jamming, Force Chains and Fragile Matter

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We consider materials whose mechanical integrity is the result of a jamming process. We argue that such media are generically “fragile”: unable to support certain types of incremental loading without plastic rearrangement. Fragility is linked to the marginal stability of force chain networks within the material. Such ideas may be relevant to jammed colloids and poured sand. The crossover from fragile (when particles are rigid) to elastoplastic behavior is explored.

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Consider a concentrated colloidal suspension of hard particles under shear (Fig. 1.(a)). Above a certain threshold of stress, this system will jam [1]. (To observe such an effect, stir a concentrated suspension of corn-starch with a spoon.) In this Letter, we propose some simple models of jammed systems like this, whose solidity stems directly from the applied stress itself. Such systems may be fundamentally different, in their mechanics, from other classes of material, such as elastic or elastoplastic solids.

In colloids, jamming apparently occurs because the particles form “force chains” along the compressional direction [1]. Even for spherical particles the lubrication films cannot prevent contacts; once these arise, an array or network of force chains can, in principle, support the shear stress indefinitely. (Brownian motion is neglected here and below.) By this definition, the material is solid.

A simple model of a force chain assumes a linear string of rigid particles in point contact. Crucially, this object can support only tangential loads [2] (Fig.2.(a)): successive contacts must be colinear, with the forces along the line of contacts, to prevent torques on particles within the chain. (Friction at the contacts does not obviate this requirement, nor does particle asphericity.)

Let us model a jammed colloid by an assembly of such force chains, characterized by a director  $\mathbf{n}$ , in a sea of “spectator” particles, and incompressible solvent. (We ignore for the moment any “collisions” between force chains or deflections caused by weak interaction with the spectators.) In static equilibrium, with no body forces acting, the compressive stress tensor is then

$$\sigma_{ij} = P\delta_{ij} + \Lambda n_i n_j \quad (1)$$

where  $P$  is an isotropic pressure, and  $\Lambda$  a compressive stress carried by the force chains.

Eq.(1) permits static equilibrium only so long as the applied compression is along  $\mathbf{n}$ ; while this remains true, small, or even large, incremental loads can be accommodated reversibly, by what is (ultimately) an elastic mechanism. But the material is certainly not an elastic solid, for if instead one tries to shear the sample in a slightly different direction (causing a rotation of the principal stress axes) static equilibrium cannot be maintained

without changing the director  $\mathbf{n}$ . And since  $\mathbf{n}$  describes force chains that pick their ways through a dense sea of spectator particles, it cannot simply rotate; instead, the existing force chains must be abandoned and new ones created with a slightly different orientation. This entails dissipative, plastic, reorganization, during which the system will re-jam to support the new load.

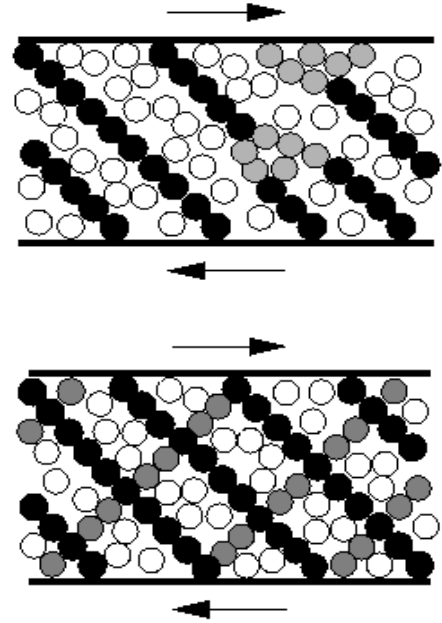


FIG. 1. (a) A jammed colloid (schematic). Black: force chains; grey: other force-bearing particles; white: spectators. (b) Idealized rectangular network of force chains.

The jammed colloid is an example of *fragile matter*. The medium can statically support applied shear stresses (within some range), but does so by virtue of a self-organized internal structure, whose mechanical properties have evolved to support the load itself. Its incremental response can be elastic only to *compatible* loads; incompatible loads (in this case, those with a different compression axis), even if small, will cause finite, plastic reorganizations. The system resembles a liquid crystal,

except that incompatible loads cause transient rearrangement, not steady flow. An inability to elastically support *some* infinitesimal loads is our definition of “fragility”. This extends naturally to small perturbations of other types, such as changes in temperature, which can lead to “static avalanches” of rearrangements [3].

Jamming may lead *generically* to fragile matter. If a system arrests as soon as it can support the external load, its state is likely to be only marginally stable. Incompatible perturbations induce rearrangements, leaving the system in a newly jammed, equally fragile, state. This scenario is related to suggestions that rigidity emerges by successive buckling of force chains (impeded by spectators) in glasses and granular matter [4]. It also resembles self-organized criticality (SOC) [5]; we return to this later (see also [3,6]). However, we focus first on simple models of the fragile state in static equilibrium.

Consider again the (homogeneously) jammed colloid. What body forces can it now support *without* plastic rotation of the director? Various models are possible. One is to assume that Eq.(1) continues to apply, with  $P(\mathbf{r})$  and  $\Lambda(\mathbf{r})$  now varying in space. If  $P$  is simply a fluid pressure, a localized body force can be supported only if it acts along  $\mathbf{n}$ . Thus (as in a bulk fluid) no static Green function exists for a general body force. To support general loadings in three dimensions in fact requires more than one orientation of force chain, perhaps forming a network or skeleton [7–10]. A simple model capable of describing this is:

$$\sigma_{ij} = \Lambda_1 n_i n_j + \Lambda_2 m_i m_j + \Lambda_3 l_i l_j \quad (2)$$

The directors  $\mathbf{n}, \mathbf{m}, \mathbf{l}$  describe three nonparallel populations of force chains and can also be seen as *characteristics* along which the static Green function propagates [12,15]. The  $\Lambda$ ’s are compressive pressures acting along these. Body forces cause  $\Lambda_{1,2,3}$  to vary in space.

Based on the above, we can distinguish different levels of fragility, according to whether incompatible loads include localized body forces (“bulk” fragility), or are limited to forces acting at the boundary (“boundary” fragility). (With Eq.(2), for example, a body force can always be transmitted to the boundary but such boundary forces cannot then be specified independently – see below.) In disordered systems one might also distinguish between macro-fragile responses involving changes in the *mean* orientation of force chains, and the micro-fragile responses of individual contacts. Here we focus on macro-fragility; but if the medium is susceptible to long-ranged ‘static avalanches’ [3] the distinction may be blurred.

Returning to the model of Eq.(2), the chosen values of the three directors (two in 2-d) clearly should depend on how the system came to be jammed (its “construction history”). If it jammed in response to a constant stress, switched on suddenly at some earlier time, one can argue that the history is specified *purely by the stress tensor itself* (unless body forces dominate). In this case, if one director points along the major compression axis, then

by symmetry any others should lie at rightangles to it (Fig. 1.(b)). Applying a similar argument to the intermediate axis leads to the ansatz that all three directors lie along principal stress axes; this is perhaps the simplest model in 3-d [11]. (One version of this argument links force chains with the fabric tensor [9], which is itself typically coaxial with the stress [10].) If so, Eq.(2) does not change form if an arbitrary isotropic pressure field  $P$  is added. With perpendicular directors as just described, Eq.(2) becomes the “fixed principle axes” (FPA) model. We proposed this recently to describe stress propagation in conical piles of sand, constructed by pouring cohesionless grains from a point source onto a rough rigid support. This model accounts quite well for the forces measured experimentally beneath conical sandpiles [14,12,6]. More generally, Eq.(2) can be obtained by assuming a linear closure relation between the components of the stress tensor [12].

Although the formation of dry granular aggregates under gravity is not normally described in terms of jamming, it is closely related. Indeed, the filling of silos and the motion of a piston in a cylinder filled with grains exhibits jamming and stick-slip phenomena related directly to formation of force chains in these geometries; see [13]. Hence fragile models of granular media must merit serious consideration. They share some features with recent *hypoplastic* models of such media [9]. Granular matter has, however, traditionally been described by various forms of elastoplasticity, which we now compare and contrast with the above idea of fragility.

The characteristics evident in Eq.(2) are directly related to the *hyperbolic* nature of the differential equations governing stress propagation in fragile packings [15,12,6]. With a zero-force boundary condition at the upper surface of a pile [6], this gives a well-posed problem: the forces acting at the base follow uniquely from the body forces by integration: the Green function (in 2-d) comprises two rays connecting a source to the base [15,12]. (Closely analogous remarks apply in three dimensions.) The same does not hold [16] for traditional elastoplastic continuum models [17] whose equations are (in simple cases) elliptic in elastic zones and hyperbolic in plastic ones. In the sandpile, where an elastic zone contacts the support, the forces acting at the base cannot then be found without specifying a displacement field there. But such a displacement field has no clear physical meaning for a sandpile created by pouring. To define it, one requires a *reference state* in which the stresses (gravity) are removed. Such a state is undefined, just as it is undefined for a jammed colloid which, in the absence of the applied shear stress, is a fluid.

This ‘elastic indeterminacy’ of sandpiles has no facile resolution [16]. Suggestively, the underlying problem (absence of a reference state) arises *precisely when* elastoplasticity might give way to fragility: in systems whose solidity arises because of the load itself. Nonetheless, a crossover between fragile and elastoplastic descriptions may exist, at least in the context of very small incremen-

tal loads (for which the reference state can be defined in terms of a pre-existing, gravitationally loaded pile). For example, one *might* expect that, for poured sand, sound waves of sufficiently small amplitude could propagate normally (although in fact this is far from obvious experimentally [19]). Likewise in our jammed colloid, *extremely small* rotations of the principal axes might be accommodated by an elastic, and not a plastic, mechanism.

We next show, for a specific example of a fragile granular skeleton, that just such a crossover can arise from a slight particle deformability. We consider a highly idealized, 2-d rectangular skeleton of rigid particles, Fig.1.(b). In this material, where the tangential compressive forces balance at each node [20], the shear stress must vanish across planes parallel to  $\mathbf{n}$  and  $\mathbf{m}$  (that is,  $\sigma_{nm} = \sigma_{mn} = 0$ ). For simplicity we also assume that the ratio  $\Lambda_1/\Lambda_2$  (and its inverse) cannot exceed some constant  $K$  (for example to avoid buckling of the stress paths). This implies a Coulomb-like inequality,  $|\sigma_{pq}| \leq \sigma_{qq} \tan \phi$ , for all other orthogonal unit vector pairs  $\mathbf{q}, \mathbf{p}$ ; here  $\tan \phi$  is a material constant such that  $K = (1 - \sin \phi)/(1 + \sin \phi)$ .

Next a small degree of particle deformability is introduced. This relaxes slightly the colinearity requirement of forces along chains, because the point contacts between particles are now flattened (Fig.2.(b)). Clearly the ratio  $\epsilon$  of the maximum transverse load to the normal one will vanish in some specified way (dependent on contact geometry) with the mean particle deformation. The same ratio  $\epsilon$  defines, in effect, the maximum elastic angular deviation of the force chains. The system can thus be described as an (anisotropic) elastic body subject to a yield criterion of the following form:

$$|\sigma_{pq}| \leq \sigma_{qq} \tan \Phi(\mathbf{q} \cdot \mathbf{n}) \quad (3)$$

where  $\Phi(x)$  is a smooth function that is small (of order  $\epsilon$ ) in a narrow range (of order  $\epsilon$  wide) of orientations around  $x = 0$  (and  $x = 1$ ), but close to  $\phi$  outside this interval.

For finite  $\epsilon$  this material will have mixed elliptic/hyperbolic equations of the usual elastoplastic type. But the resulting elastic and plastic zones must somehow arrange themselves so as to obey the FPA model to within terms that vanish as  $\epsilon \rightarrow 0$ . If, in a sandpile,  $\epsilon$  is small but finite, then stresses will depend on the detailed boundary conditions at the base of the pile, but only through small corrections to the leading (FPA) result. These deviations can accommodate an elastic response to very small incremental loads (on a scale set by  $\epsilon$ ). But for the macroscopic stress pattern to differ significantly from the hyperbolic prediction, one requires *appreciable particle deformation*. When the mean stresses are large enough to cause this ( $\epsilon \simeq 1$ ), “ordinary” elastic or elastoplastic behavior will be recovered. Conversely, the fragile, hyperbolic limit emerges as *the limit of high particle rigidity* for this simplified model skeleton. Thus fragile models of granular or jammed matter, properly interpreted, need not contradict (though equally they do not require) an underlying elastoplastic description.

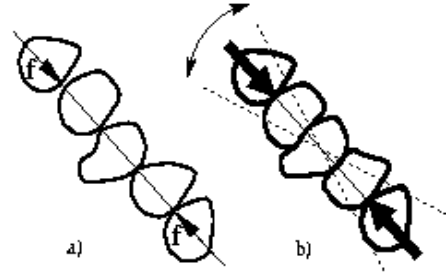


FIG. 2. (a) A force chain of hard particles (any shape) can statically support only tangential compression. (Body forces acting directly on these particles are neglected.) (b) Finite deformability allows small transverse loads to arise.

How valid are these ideas for granular media? The existence of tenuous force-chain skeleton is hardly in doubt [7,22,8–10]. Simulations of frictional spheres show most of the deviatoric stress to arise from force chains, while tangential interparticle forces and “spectator” contacts contribute mainly an isotropic pressure [8,10]. (Of course the specific geometry of Fig.1.(b) is grossly oversimplified: although the force chains are anisotropic, they are not very straight and have frequent collisions [21,8].) Are such skeletons actually fragile, as our models suggest, or do they have an appreciable range of incremental elastic response? And, for sand under gravity, does a fragile or an elastoplastic model better describe its (nonperturbative) response to gravity itself?

We believe that granular media are often close to the fragile limit. Firstly, we return to an earlier argument: in systems whose solidity arises by a jamming process, fragility may arise generically if the material arrests in states that can only just support the applied load. This may be a reasonable picture for sandpiles created by pouring. It could also apply to unconsolidated dry grains in various other geometries.

Second, the probability distribution for interparticle forces  $p(f)$  does not vanish at zero force [22]. This is consistent with the idea that a slight, incompatible change in load (relative scale  $\delta/\bar{f}$  with  $\bar{f}$  the mean interparticle force) can induce a fraction  $p(0)\delta$  of contacts to switch from spectator type ( $f \simeq 0$ ) to force-chain type ( $f \simeq \bar{f}$ ). The effect of such rearrangements would then be comparable with the elastic response ( $\bar{f} \rightarrow \bar{f} \pm \delta$ ), and so formally destroy the elastic regime; we expect this effect to be amplified by any long-range rearrangement of force chains that a local contact change may induce [3]. Third, simulations do indeed show strong rearrangement of the skeleton under small changes of compression axis; the skeleton is indeed “self-organized” [8,10]. There is also evidence in some instances for internal cascades of rearrangement [3,13] in response to small disturbances. Although the latter is strongly reminiscent of SOC [5], our simplified models show that fragility is a rather different concept which can arise, at least in principle, in regular geometries, and without the self-similarity in the

response to a small perturbation that characterizes SOC. In fact SOC-like concepts underly recent discussions of dynamic attractors in hypoplastic models [9], and are not far removed from the (much older) critical state theories of soil mechanics [18]. The latter primarily address *dilatancy*: the tendency of dense granular media to expand upon shearing. Jamming can be viewed as the constant-volume counterpart of this process.

We await further experimental guidance on the extent to which granular materials are, in practice, fragile. Various experimental tests of specific fragile models are suggested elsewhere [12,6,3]; these predict anomalies in both correlation and response functions. More generally, the negligibility of any incremental elastic range (as postulated in fragile models for incompatible loads) can be probed by various experiments including sound transmission. The latter do indeed show very peculiar behaviour [19], possibly related to the fact that the sound wave itself causes rearrangements. In addition, computer simulations should clarify the relationship between fragility and the extreme nonlinearity which, for cohesionless dry sand, enters because tensile contact forces are forbidden. Only when the probability density  $p(0)$  of zero contact force becomes small can this be safely ignored; and this might not arise before strong particle deformation occurs ( $\epsilon \simeq 1$ ).

More generally, other candidates for fragile matter include jammed colloids, weak particulate gels, and flow-induced defect textures in liquid crystals, all of which can self-organize so as to support an applied stress.

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Note added: we also refer the reader to C. Moukarzel, cond-mat/9803120, where somewhat related ideas can be found.

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- [1] R. S. Farr, J. R. Melrose, *Physical Review E*, **55**, 7203 (1997); R. C. Ball, J. R. Melrose, *Adv. Colloid Interface Sci.*, **59**, 19 (1995); J. R. Melrose, R. C. Ball, *Europhys. Lett.*, **32**, 535 (1995).
  - [2] S. F. Edwards and R. B. Oakeshott, *Physica D*, **38**, 88 (1989); S. F. Edwards, C. C. Mounfield, *Physica A*, **226**, 1, (1996).
  - [3] P. Claudin, J.-P. Bouchaud, M. E. Cates and J. P. Wittmer, *Phys. Rev. E*, April 1998, in press, cond-mat/9710100. P. Claudin and J.-P. Bouchaud, *Phys. Rev. Lett.* **78**, 231 (1997).
  - [4] S. Alexander, *Phys. Reports*, in press; P. S. Cundall and O. D. L. Strack, in *Mechanics of Granular Materials: New Models and Constitutive Relations*, J. T. Jenkins and M. Satakem Eds, Elsevier 1983.
  - [5] P. Bak, *How Nature Works; The Science of Self-Organized Criticality* (Oxford University Press, Oxford 1997). Here SOC-like ideas enter at the level of the *statics*, not avalanche dynamics (as studied in the SOC literature).
  - [6] J.-P. Bouchaud, P. Claudin, M. E. Cates, J. P. Wittmer, *Physics of Dry Granular Media, Cargese 1997*, H.J. Herrmann *et al.*, Eds., Kluwer in press; cond-mat/9711135.
  - [7] P. Dantu, *Annales des Ponts et Chaussées*, **4**, 144 (1967).
  - [8] C. Thornton, in *KONA Powder and Particle* **15** (1997); C. Thornton and G. Sun, in *Numerical Methods in Geotechnical Engineering*, I. Smith, Ed., Balkema (1994).
  - [9] D. Kolymbas, *Archive of Appl. Mech.*, **61**, 143 (1991); G. Gudehus, in Ref. [17].
  - [10] F. Radjai, D. E. Wolf, M. Jean, J. J. Moreau, *Phys. Rev. Lett.* **90**, 61 (1998), and references therein.
  - [11] A less symmetrical model allows all three directors in Eq.(2) to be oblique. For example, in a sandpile the construction history does not solely involve the stress tensor, but also, e.g., the free surface orientation. See [12].
  - [12] J. P. Wittmer, P. Claudin, M. E. Cates and J.-P. Bouchaud, *Nature*, **382**, 336 (1996); J. P. Wittmer, P. Claudin, M. E. Cates, *J. Physique I*, **7**, 39 (1997).
  - [13] E. Kolb, T. Mazozi, J. Duran, E. Clement, in preparation. P. Claudin, J.-P. Bouchaud, cond-mat/9801057, to appear in *Granular Matter*.
  - [14] Smid, J. and Novosad, J. in *Proc. of 1981 Powtech Conference, Ind. Chem. Eng. Symp.* **63**, D3V 1-12 (1981); R. Brockbank, J. M. Huntley and R. C. Ball, *J. Physique II*, **7**, 1521 (1997).
  - [15] J.-P. Bouchaud, M. E. Cates, and P. Claudin, *J. Physique I*, **5** 639 (1995).
  - [16] M. E. Cates, J. P. Wittmer, J.-P. Bouchaud, and P. Claudin, *Phil. Trans. Roy. Soc. Lond. Ser. A.*, submitted. cond-mat/9803135.
  - [17] See, e.g., *Proc. 3rd Int. Conf. on Powders and Grains, Durham NC 18-23 May 1997*, R. P. Behringer and J. T. Jenkins, Eds. Balkema, Rotterdam 1997, especially F. Cantelaube and J. D. Goddard, pp. 185-194.
  - [18] D. Muir Wood, *Soil Behaviour and Critical State Soil Mechanics* Cambridge University Press, 1990.
  - [19] C.-h. Liu, S. R. Nagel, *Phys. Rev. Lett.* **68**, 2301 (1992); C.-h. Liu, S. R. Nagel, *Phys. Rev. B* **48**, 15646 (1992).
  - [20] The tangential force rule for force chains ensures an equivalence between ordered arrays of these and packings of frictionless spheres. (R. C. Ball and S. F. Edwards, in preparation; [15], and references therein.)
  - [21] Collisions between force chains give the impression of a three-fold coordinated network in two dimensions (see [8]). Such a network is “bulk fragile” in contrast to Eq.2. However, individual collision points might be micro-fragile, with changes in boundary load still required to induce macroscopic rearrangement.
  - [22] C. H. Liu, S. R. Nagel, D. A. Schecter, S. N. Copper-smith, S. Majumdar and T. A. Witten, *Science* **269** 513 (1995); S. R. Nagel, *Rev. Mod. Phys.*, **64**, 321 (1992).