DIMENSIONAL ANALYSIS

1. Introduction

Dimensional analysis is a technique that treats the general forms of equations governing natural phenomena. It provides procedures of judicious grouping of variables associated with a physical phenomenon to form dimensionless products of these variables; therefore, without destroying the generality of the relationship, the equation describing the physical phenomenon may be more easily determined experimentally. It guides the experimenter in the selection of experiments capable of yielding significant information and in the avoidance of redundant experiments, and makes possible the use of scale models for experiments (see also DESIGN OF EXPERIMENTS). The method is particularly valuable when the problems involve a large number of variables. On such occasions, dimensional analysis may reveal that, whatever the form of the inaccessible final solution, certain features of it are obligatory. The technique has been utilized effectively in engineering modeling (1–7).

The method of dimensional analysis is not new. It can be traced to Newton, who at the time was laying the foundation of mechanics as a fundamental branch of science. The validity of the method is based on the premise that any equation that correctly describes a physical phenomenon must be dimensionally homogeneous. This principle of dimensional homogeneity, which states in effect that quantities of different kinds cannot be added together, is of fundamental importance in dimensional analysis, and was first expressed by J. Fourier's classic work *La Théorie Analytique de la Chaleur*, published in 1822. He not only introduced the notion of dimensional homogeneity but also the conception of what is today termed the dimensional formula. In 1914, Buckingham (8) made a significant contribution with his famous pi theorem, which possibly prompted Lord Rayleigh (9) shortly thereafter to observe, "It happens not infrequently that results in the form of 'laws' are put forward as novelties on the basis of elaborate experiments which might have been predicted *a priori* after a few minutes' consideration". Needless to say, Fourier's and Buckingham's works have been used, elaborated upon, and extended by many others (10–38). A measure of this interest is reflected in three comprehensive bibliographies (10–12) in which >600 research contributions are referenced.

2. Units and Dimensions

The concepts used to describe natural phenomena are based on the precise measurement of quantities. The quantitative measure of anything is a number that is found by comparing one magnitude with another of the same type. It is necessary to specify the magnitude of the quantity used in making the comparison if the number is to be meaningful. The statement that "the length of a car is 6 meters" implies that a length has been chosen, namely, 1 m, and that the ratio of the length of the car to the chosen length is 6. The chosen magnitudes, such as the meter, are called *units* of measurement. The result of a measurement is represented by a number followed by the name of the unit that was used in making the measurement (see UNITS). The statement that the area of a room is 30 m^2 indicates that the unit of measurement is 1 m^2. Thus, to each kind of physical quantity there corresponds an appropriate kind of unit. The physical concepts such as length, area, and time are referred to as *dimensions*, which are different from units. The length of a car is 6 m, which is equivalent to 19.7 ft or 6.56 yard.

Classical physics is built on the foundation of the laws of motion. It was felt at the time that the entire subject could be based on the laws of classical mechanics, and further work would undoubtedly make electromagnetism another branch of mechanics. Under these circumstances, it was natural to regard length l, mass m, and time t as the fundamental, primary, or reference dimensions. However, such designations lead to dimensional ambiguity in that two distinct concepts may possess the same dimensions. The system works fairy well for mechanics. The most notable ambiguity occurs with energy and torque. However, in electromagnetism the situation is bad. The classical arrangement employs electrostatic and electromagnetic systems, and the same concept may lead to different dimensions. For example, in the electrostatic system capacitance and length have the same dimensions, and in the electromagnetic system inductance and length are not dimensionally distinguished. On the subject of heat, ambiguities occur between entropy and mass (13), or temperature and reciprocal length (14), depending on the assumptions made about the dimensionality of some constants. This does not mean that dimensional ambiguity is a fact of nature; it merely shows an imperfection in our human scheme of assigning primary concepts.

Over the years, the number of reference dimensions in physics has evolved from the original three, to four, to five, and then gradually downward to an absolutely necessary one, and then upward again through an understanding that, though only one is absolutely necessary, a considerable convenience can stem from using three, to four, or five reference dimensions depending on the problem at hand (1, 6, 7, 15–20). There is nothing sacrosanct about the number of reference dimensions, and dimensional analysis is merely a tool that may be manipulated at will (1). This principle of free choice of the reference dimensions has been widely accepted, although one still finds references to true dimensions. Thus, an important step in dimensional analysis is the selection of reference dimensions in such a way that the others, called the secondary or derived dimensions, can be expressed in terms of them. The relation between reference and derived dimensions is generally established either through the fundamental law or equation governing the phenomenon or through definitions. When length, mass, and time are taken to be the reference dimensions, the dimensions of velocity v, eg, are the dimensions of length divided by time, or expressed by symbols, $v = lt^{-1}$. Likewise, Newton's law of motion relates force, mass, and acceleration by

$$\text{force} = \text{constant} \times \text{mass} \times \text{acceleration} \tag{1}$$

The dimensions of force f must be $(\text{mass} \cdot \text{length})/\text{time}^2$ or $f = mlt^{-2}$. The expressions like $v = lt^{-1}$ and $f = mlt^{-2}$ are referred to as dimensional formulas. The exponents of dimensions of a physical quantity are the powers of the reference dimensions in which it is expressed. Thus, the exponents of the dimensions of

Table 1. **Exponents of Dimensions for Mechanical Quantities in Absolute, Gravitational, and Engineering Systems**

	Absolute			Gravitational			Engineering			
Quantity	m	l	t	f	l	t	f	m	l	t
acceleration	0	1	−2	0	1	−2	0	0	1	−2
angular acceleration	0	0	−2	0	0	−2	0	0	0	−2
angular velocity	0	0	−1	0	0	−1	0	0	0	−1
area	0	2	0	0	2	0	0	0	2	0
angular momentum	1	2	−1	1	1	1	0	1	2	−1
density, mass	1	−3	0	1	−4	2	0	1	−3	0
energy, work	1	2	−2	1	1	0	1	0	1	0
force	1	1	−2	1	0	0	1	0	0	0
frequency	0	0	−1	0	0	−1	0	0	0	−1
length	0	1	0	0	1	0	0	0	1	0
linear acceleration	0	1	−2	0	1	−2	0	0	1	−2
linear momentum	1	1	−1	1	0	1	0	1	1	−1
linear velocity	0	1	−1	0	1	−1	0	0	1	−1
mass	1	0	0	1	−1	2	0	1	0	0
moment of inertia	1	2	0	1	1	2	0	1	2	0
power	1	2	−3	1	1	−1	1	0	1	−1
pressure	1	−1	−2	1	−2	0	1	0	−2	0
stress	1	−1	−2	1	−2	0	1	0	−2	0
surface tension	1	0	−2	1	−1	0	1	0	−1	0
time	0	0	1	0	0	1	0	0	0	1
viscosity, absolute	1	−1	−1	1	−2	1	1	0	−2	1
viscosity, kinematic	0	2	−1	0	2	−1	0	0	2	−1
volume	0	3	0	0	3	0	0	0	3	0

force are 1 in mass, 1 in length, and −2 in time. If force, length, and time are chosen as the reference dimensions, then mass becomes secondary. In either of these two choices, the constant in Newton's law is dimensionless. However, if force, mass, length, and time are chosen as the reference dimensions, the constant is no longer dimensionless and the units are generally selected so that the constant is numerically equal to the standard acceleration of gravity. Table 1 lists the exponents of dimensions of some common variables in mechanics with respect to these three choices of reference dimensions [(m,l,t); (f,l,t); (f,m,l,t)], which give rise to the absolute, gravitational, and engineering systems of dimensions, respectively.

To eliminate the ambiguities in the subject of electricity and magnetism, it is convenient to add charge q to the traditional l, m, and t dimensions of mechanics to form the reference dimensions. In many situations, permittivity ϵ or permeability μ is used in lieu of charge. For thermal problems, temperature T is considered as a reference dimension. Tables 2 and 3 list the exponents of dimensions of some common variables in the fields of electromagnetism and heat.

Other dimensional systems have been developed for special applications which can be found in the technical literature. In fact, to increase the power of dimensional analysis, it is advantageous to differentiate between the lengths in radial and tangential directions (13). In doing so, ambiguities for the concepts of energy and torque, as well as for normal stress and shear stress, are eliminated (see Ref. 13).

Table 2. **Exponents of Dimensions for Electromagnetic Quantities**

Quantity	l	m	t	q	l	m	t	ϵ	l	m	t	μ
charge	0	0	0	1	3/2	1/2	−1	1/2	1/2	1/2	0	−1/2
capacitance	−2	−1	2	2	1	0	0	1	−1	0	2	−1
current	0	0	−1	1	3/2	1/2	−2	1/2	1/2	1/2	−1	−1/2
electric field intensity	1	1	−2	−1	−1/2	1/2	−1	−1/2	1/2	1/2	−2	1/2
electric potential difference	2	1	−2	−1	1/2	1/2	−1	−1/2	3/2	1/2	−2	1/2
electric flux	0	0	0	1	3/2	1/2	−1	1/2	1/2	1/2	0	−1/2
electric flux density	−2	0	0	1	−1/2	1/2	−1	1/2	−3/2	1/2	0	−1/2
inductance	2	1	0	−2	−1	0	2	−1	1	0	0	1
magnetic field intensity	−1	0	−1	1	1/2	1/2	−2	1/2	−1/2	1/2	−1	−1/2
magnetic flux	2	1	−1	−1	1/2	1/2	0	−1/2	3/2	1/2	−1	1/2
magnetic flux density	0	1	−1	−1	−3/2	1/2	0	−1/2	−1/2	1/2	−1	1/2
magnetomotive force	0	0	−1	1	3/2	1/2	−2	1/2	1/2	1/2	−1	−1/2
permeability	1	1	0	−2	−2	0	2	−1	0	0	0	1
permittivity	−3	−1	2	2	0	0	0	1	−2	0	2	−1
resistance	2	1	−1	−2	−1	0	1	−1	0	0	−1	1

Table 3. **Exponents of Dimensions for Thermal Quantities**

Quantity	l	m	t	T	f	l	t	T
coefficient of thermal expansion	0	0	0	−1	0	0	0	−1
entropy	2	1	−2	−1	1	1	0	−1
temperature	0	0	0	1	0	0	0	1
thermal energy (heat)	2	1	−2	0	1	1	0	0
thermal power	2	1	−3	0	1	1	−1	0
thermal conductivity	1	1	−3	−1	1	0	−1	−1
thermittivity	2	0	−2	−1	0	2	−2	−1

3. Dimensional Matrix and Dimensionless Products

An appropriate set of independent reference dimensions may be chosen so that the dimensions of each of the variables involved in a physical phenomenon can be expressed in terms of these reference dimensions. In order to utilize the algebraic approach to dimensional analysis, it is convenient to display the dimensions of the variables by a matrix. The matrix is referred to as the dimensional matrix of the variables and is denoted by the symbol $\boldsymbol{D}$. Each column of $\boldsymbol{D}$ represents a variable under consideration, and each row of $\boldsymbol{D}$ represents a reference dimension. The ith row and jth column element of $\boldsymbol{D}$ denotes the exponent of the reference dimension corresponding to the ith row of $\boldsymbol{D}$ in the dimensional formula of the variable corresponding to the jth column. As an illustration, consider Newton's law of motion, which relates force F, mass M, and acceleration A by (eq. 2):

$$F = \text{constant} \times MA \tag{2}$$

If length l, mass m, and time t are chosen as the reference dimensions, from Table 1 the dimensional formulas for the variables F, M, and A are as follows:

Variables	Dimensional formulas
F	$m^1 l^1 t^{-2}$
M	$m^1 l^0 t^0$
A	$m^0 l^1 t^{-2}$

The dimensional matrix associated with Newton's law of motion is obtained as (eq. 3)

$$D = \begin{array}{c} \\ m \\ l \\ t \end{array} \begin{array}{c} \begin{array}{ccc} F & M & A \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -2 & 0 & -2 \end{bmatrix} \end{array} \tag{3}$$

In this example, the exponents of dimensions in the dimensional formula of the variable F are 1, 1, and -2, and hence the first column is $(1,1, -2)$. Likewise, the second and third columns of $\boldsymbol{D}$ correspond to the exponents of dimensions in the dimensional formulas of the variables M and A, respectively.

As indicated earlier, the validity of the method of dimensional analysis is based on the premise that any equation that correctly describes a physical phenomenon must be dimensionally homogeneous. An equation is said to be dimensionally homogeneous if each term has the same exponents of dimensions. Such an equation is of course independent of the systems of units employed provided the units are compatible with the dimensional system of the equation. It is convenient to represent the exponents of dimensions of a variable by a column vector called dimensional vector represented by the column corresponding to the variable in the dimensional matrix. In equation 3, the dimensional vector of force F is $[1,1, -2]'$ where the prime denotes the matrix transpose.

Suppose that there are n variables $Q_1, Q_2, \ldots, Q_n$ that are involved in a physical phenomenon whose dimensional vectors are $\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots, \boldsymbol{D}_n$, respectively. This phenomenon can generally be expressed by (eq. 4):

$$f(Q_1, Q_2, \ldots, Q_n) = 0 \tag{4}$$

When such a function is established or assumed, it will still exist even after the variables are intermultiplied in any manner whatsoever. This means that each variable in the equation can be combined with other variables of the equation to form dimensionless products whose dimensional vectors are the zero vector. Equation 4 can then be transformed into the nondimensional form as (eq. 5):

$$f(\pi_1, \pi_2, \ldots, \pi_n) = 0 \tag{5}$$

where the dimensionless products π_i ($i = 1, 2, \ldots, n$) can generally be expressed as the power products of the form (eq. 6):

$$\pi_i = Q_1^{x_{1i}} Q_2^{x_{2i}} \ldots Q_n^{x_{ni}} \tag{6}$$

Let $\mathrm{R}_1, \mathrm{R}_2, \ldots, \mathrm{R}_m$ be a set of chosen reference dimensions. Then the dimensional formulas for the variables $\boldsymbol{Q}_i$ are given by (eq. 7):

$$\mathrm{R}_1^{\mathrm{d}_{1i}} \mathrm{R}_2^{\mathrm{d}_{2i}} \ldots \mathrm{R}_m^{\mathrm{d}_{mi}} \tag{7}$$

where the exponents of dimensions are represented by the dimensional vectors as (eq. 8):

$$\boldsymbol{D}_i' = [\mathrm{d}_{1i}\, \mathrm{d}_{2i} \ldots \mathrm{d}_{mi}] \qquad i = 1, 2, \ldots, n \tag{8}$$

By using eq. 7, the dimensional formulas for π_i of equation 6 can be written to give (eq. 9):

$$\left[\mathrm{R}_1^{\mathrm{d}_{11}} \mathrm{R}_2^{\mathrm{d}_{21}} \ldots \mathrm{R}_m^{\mathrm{d}_{m1}}\right]^{x_{1i}} \left[\mathrm{R}_1^{\mathrm{d}_{12}} \mathrm{R}_2^{\mathrm{d}_{22}} \ldots \mathrm{R}_m^{\mathrm{d}_{m2}}\right]^{x_{2i}} \ldots \left[\mathrm{R}_1^{\mathrm{d}_{1n}} \mathrm{R}_2^{\mathrm{d}_{2n}} \ldots \mathrm{R}_m^{\mathrm{d}_{mn}}\right]^{x_{ni}} \tag{9}$$

Since π_i are dimensionless products having dimensional vectors equal to the zero vector, the exponents of the R_j ($j = 1, 2, \ldots, m$) must add up to zero, giving (eq. 10):

$$\begin{aligned} \mathrm{d}_{11}x_{1i} + \mathrm{d}_{12}x_{2i} + \cdots + \mathrm{d}_{1n}x_{ni} &= 0 \\ \mathrm{d}_{21}x_{1i} + \mathrm{d}_{22}x_{2i} + \cdots + \mathrm{d}_{2n}x_{ni} &= 0 \\ &\vdots \\ \mathrm{d}_{m1}x_{1i} + \mathrm{d}_{m2}x_{2i} + \cdots + \mathrm{d}_{mn}x_{ni} &= 0 \end{aligned} \tag{10}$$

In terms of the dimensional vectors of equation 8, equation 10 can be written as (eqs. 11–13):

$$[\boldsymbol{D}_1 \boldsymbol{D}_2 \cdots \boldsymbol{D}_n] X_i = 0, \qquad i = 1, 2, \ldots, n \tag{11}$$

where

$$X_i' = [x_{1i}\, x_{2i} \ldots x_{ni}] \tag{12}$$

or more compactly

$$\boldsymbol{DX} = 0 \tag{13}$$

where $X = X_i$, $i = 1, 2, \ldots, n$. Thus, the product of a set of variables is dimensionless if, and only if, the exponents of these variables are a solution of the homogeneous linear algebraic equation (13). A vector $\boldsymbol{X}$ is said to be a B-vector of $\boldsymbol{D}$ if it is a solution of equation 13. The corresponding dimensionless product associated with the variables of a B-vector is called a B-number (21,22). Frequently, the term pi number is also used by many authors because it was first introduced by Buckingham (8) in 1914 who used the symbol π for a dimensionless product or group. In fact, the term pi was even attached to his contributions to dimensional analysis, and is known as Buckingham's pi theorem. But this usage is deprecated because of possible confusion with the universal constant of $\pi = 3.14159$. Therefore, the choice of his initial B is preferred to that of the term π.

The following example illustrates the above procedure:

Example 1. The problem is to find the period P of a simple pendulum swinging in a vacuum under the influence of gravity. To write an equation for the period, the first step is to consider what physical quantities affect the period, which requires some prior knowledge upon which the intuitive judgment can be based. On this basis, it is clear that the period depends on the mass M of the bob, the length L of the string supporting the bob, and of course, the acceleration g owing to the force of gravity. As before, mass m, length l, and time t are chosen as the reference dimensions, ie, $R_1 = m$, $R_2 = l$, and $R_3 = t$. From Table 1, the dimensional formulas for the variables M, L, P, and g, together with their dimensional vectors are as shown below.

Variables	Dimensional formulas	Dimensional vectors
$Q_1 = M$	$m^1 l^0 t^0$	$D_1' = [1, 0, 0]$
$Q_2 = L$	$m^0 l^1 t^0$	$D_2' = [0, 1, 0]$
$Q_3 = P$	$m^0 l^0 t^1$	$D_3' = [0, 0, 1]$
$Q_4 = g$	$m^0 l^1 t^{-2}$	$D_4' = [0, 1, -2]$

Thus, the dimensional formulas for π_i of equation 6 can be expressed as (eq. 14):

$$(m^1 l^0 t^0)^{x_{1i}} (m^0 l^1 t^0)^{x_{2i}} (m^0 l^0 t^1)^{x_{3i}} (m^0 l^1 t^{-2})^{x_{4i}} \tag{14}$$

whose dimensional vector must be the zero vector requiring (eq. 15):

$$\begin{array}{c c} & \begin{array}{cccc} M & L & P & g \end{array} \\ \begin{array}{c} m \\ l \\ t \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \end{array} \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \\ x_{4i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

the coefficient matrix of which is identified as the dimensional matrix $\boldsymbol{D}$ associated with the pendulum problem, and $i = 1, 2, 3, 4$. Solving x_{1i}, x_{2i}, and x_{3i} in terms of x_{4i} yields the desired B-vectors of $\boldsymbol{D}$ as

$$\boldsymbol{X}_i = [0\ -x_{4i}\ 2x_{4i}\ x_{4i}] \qquad i = 1, 2, \ldots, 4 \tag{16}$$

where $x_{4i} \neq 0$ are arbitrary constants. The corresponding B-numbers become (eq. 17).

$$\pi_i = M^0 L^{-x_{4i}} P^{2x_{4i}} g^{x_{4i}} \tag{17}$$

Since the solutions $\boldsymbol{X}_i$ are related to one another by a multiplicative constant, there is only one linearly independent solution, and hence only one independent

B-number. Choose, for simplicity, $x_{4i} = 1$. Equation 17 can be rewritten as (eq. 18):

$$P = \text{constant}\ \sqrt{L/g} \tag{18}$$

since π_i is a constant. The value of this constant, which is known to be 2π, cannot be determined by the method of dimensional analysis, and must be evaluated experimentally or analytically.

The example demonstrates that not all the B-numbers π_i of equation 5 are linearly independent. A set of linearly independent B-numbers is said to be complete if every B-number of $\boldsymbol{D}$ is a product of powers of the B-numbers of the set. To determine the number of elements in a complete set of B-numbers, it is only necessary to determine the number of linearly independent solutions of equation 13. The solution to the latter is well known and can be found in any text on matrix algebra [see, eg, (39) and (40)]. Thus the following theorems can be stated.

3.1. Theorem 1. The number of products in a complete set of B-numbers associated with a physical phenomenon is equal to $n - r$, where n is the number of variables that are involved in the phenomenon and r is the rank of the associated dimensional matrix.

This result was first discussed by Buckingham (8) and stated in its present form by Langhaar (23). It states in effect that an equation is dimensionally homogeneous if and only if it can be reduced to a relationship among a complete set of B-numbers. Buckingham's result (8) was originally stated as Theorem 2.

3.2. Theorem 2. A necessary and sufficient condition for an equation $f\,(Q_1, Q_2, \ldots, Q_n) = 0$ to be dimensionally homogeneous is that it should be reducible to the form $g\,(\pi_1, \pi_2, \ldots, \pi_p) = 0$, where the π values are a complete set of B-numbers of the $\boldsymbol{Q}$ variables.

This theorem does not specify how many products in a complete set of B-numbers can be expected from a given set of variables, but it does state that a physical phenomenon describable by n quantities can be rigorously and accurately described by a complete set of B-numbers. The number of products in a complete set of B-numbers may also be determined by another rule, which is equivalent to Theorem 1. The rule (Theorem 3) was first given by Van Driest (24).

3.3. Theorem 3. The number of products in a complete set of B-numbers is equal to the total number of variables minus the maximum number of these variables that will not form a dimensionless product.

To show the equivalence of Theorems 1 and 3, it is only necessary to demonstrate that the maximum number of the variables that will not form a dimensionless product is equal to the rank of the dimensional matrix $\boldsymbol{D}$.

In terms of linear vector space, Buckingham's theorem (Theorem 2) simply states that the null space of the dimensional matrix has a fixed dimension, and Van Driest's rule (Theorem 3) then specifies the nullity of the dimensional matrix. The problem of finding a complete set of B-numbers is equivalent to that of computing a fundamental system of solutions of equation 13 called a complete set of B-vectors. For simplicity, the matrix formed by a complete set of B-vectors will be called a complete B-matrix. It can also be demonstrated that the choice of reference dimensions does not affect the B-numbers (22).

3.4. Theorem 4. The set of B-numbers associated with a physical phenomenon is invariant with respect to the choice of the reference dimensions provided that the reference dimensions are considered independent, and that the number of these reference dimensions is not altered.

The implication of this theorem is important in that in computing a complete set of dimensionless products or B-numbers associated with a physical phenomenon, it does not matter which set of dimensions are chosen as the reference dimensions as long as they are independent and their number is not altered.

In Example 1, there are four variables that are involved in the pendulum problem. The associated dimensional matrix $\boldsymbol{D}$ is given in equation 15. Since the rank r of $\boldsymbol{D}$ is 3, according to Theorem 1 there are only $n - r = 4 - 3 = 1$ independent B-numbers, as expected.

Suppose now that force f, length l, and time t are chosen as the reference dimensions. From Table 1 the new dimensional matrix $\tilde{D}$; becomes (eq. 19)

$$\tilde{D} = \begin{array}{c} \\ f \\ l \\ t \end{array} \begin{array}{c} \begin{array}{cccc} M & L & P & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & -2 \end{bmatrix} \end{array} \tag{19}$$

The matrix $\hat{D}$ that will transform $\tilde{D}$ to $\boldsymbol{D}$ is the dimensional matrix of the variables force, length, and time with respect to the reference dimensions m, l, and t. Again from Table 1 equation 20 is obtained.

$$\hat{D} = \begin{array}{c} \\ m \\ l \\ t \end{array} \begin{array}{c} \begin{array}{ccc} f & l & t \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \end{array} \tag{20}$$

It is straightforward to confirm that $D = \hat{D}\tilde{D}$. Since $\hat{D}$ is nonsingular, $\boldsymbol{DX} = 0$ and $\tilde{D}X = 0$ are equivalent, possessing the same set of B-numbers.

In applying dimensional analysis, it is first necessary to be able to identify the variables that govern a particular physical phenomenon. The naming of the governing variables requires some prior knowledge of a particular branch of physics involved, which may include analytical studies, experimental observations, or both. Whatever the source, there must be some prior knowledge upon which a selection can be made.

4. Systematic Calculation of a Complete B-Matrix

Once the dimensional matrix has been set up and the number of products in a complete set of B-numbers is determined, a complete set of B-vectors must be computed. Next, a systematic procedure for this purpose is presented.

Let $\boldsymbol{D}$ be the dimensional matrix of order m by n associated with a set of variables of a physical phenomenon, where m is the number of chosen reference dimensions and n is the number of variables of the set. Without loss of generality, it may be assumed that $n \geq m$. Consider the augmented matrix (eq. 21):

$$[\boldsymbol{D}' \quad \boldsymbol{I}_n] \tag{21}$$

where, as before, the prime denotes the matrix transpose and $\boldsymbol{I}_n$ is the identity matrix of order n. Suppose that the rank of $\boldsymbol{D}$ is r. Then a finite sequence of elementary row operations of equation 21 yields an equivalent matrix of the following form [see, eg, Ref. (39)] (eq. 22):

$$\begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{C}_{13} \\ 0 & 0 & \boldsymbol{C}_{23} \end{bmatrix} \tag{22}$$

where $\boldsymbol{D}_{11}$ is a nonsingular upper triangular matrix of order r, and $\boldsymbol{D}_{12}$, $\boldsymbol{C}_{13}$, and $\boldsymbol{C}_{23}$ are matrices of orders $r \times (m - r)$, $r \times n$ and $(n - r) \times n$, respectively.

4.1. Theorem 5. The transpose C'_{23} of $\boldsymbol{C}_{23}$ is a complete B-matrix of equation 13.

It is advantageous if the dependent variables or the variables that can be regulated each occur in only one dimensionless product, so that a functional relationship among these dimensionless products may be most easily determined (8). For example, if a velocity is easily varied experimentally, then the velocity should occur in only one of the independent dimensionless variables (products). In other words, it is sometimes desirable to have certain specified variables, each of which occurs in one and only one of the B-vectors. Theorem 6 gives a necessary and sufficient condition for the existence of such a complete B-matrix. This result can be used to enumerate such a B-matrix without the necessity of exhausting all possibilities by linear combinations.

4.2. Theorem 6. Let $\boldsymbol{A}_1$ be a given complete B-matrix associated with a set of variables. Then there exists a complete B-matrix $\boldsymbol{A}_2$ of these variables such that certain specified variables each occur in only one of the B-vectors of $\boldsymbol{A}_2$ if, and only if, the rows corresponding to these specified variables in $\boldsymbol{A}_1$ are linearly independent.

The foregoing procedures are illustrated by Examples 2 and 3.

Example 2. A smooth spherical body of projected area A moves through a fluid of density ρ and viscosity μ with speed ν. The total drag δ encountered by the sphere is to be determined. Clearly, the total drag δ is a function of ν, A, ρ, and μ. As before, mass m, length l, and time t are chosen as the reference dimensions. From Table 1 the dimensional matrix is (eq. 23):

$$D = \begin{array}{c} \\ m \\ l \\ t \end{array} \begin{array}{c} \begin{array}{ccccc} \delta & \nu & A & \rho & \mu \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 2 & -3 & -1 \\ -2 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array} \tag{23}$$

To compute a complete B-matrix, the augmented matrix (eq. 24):

$$[\boldsymbol{D}' \quad \boldsymbol{I}_5] \tag{24}$$

is considered, which is given by (eq. 25):

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

The objective is to apply a sequence of elementary row operations (39) to equation 25 to bring it to the form of equation 22. Since the rank of $\boldsymbol{D}$ is 3, the order of the matrix $\boldsymbol{C}_{23}$ is $(n-r)\times n = 2\times 5$. The following sequence of elementary row operations will result in the desired form:

$$\begin{aligned}
&\text{new row } 4 = \text{row } 4 - \text{row } 1 \equiv \text{(designated as) row } 4' \\
&\text{new row } 5 = \text{row } 5 - \text{row } 1 \equiv \text{row } 5' \\
&\text{new row } 3 = \text{row } 3 - 2\times\text{row } 2 \equiv \text{row } 3' \\
&\text{new row } 4 = \text{row } 4' + 4\times\text{row } 2 \equiv \text{row } 4'' \\
&\text{new row } 5 = \text{row } 5' + 2\times\text{row } 2 \equiv \text{row } 5'' \\
&\text{new row } 4 = \text{row } 3' + \text{row } 4'' \equiv \text{row } 4''' \\
&\text{new row } 5 = \text{row } 5'' + 0.5\times\text{row } 3' \equiv 5'''
\end{aligned}$$

The corresponding matrix in partitioned form is given by (eq. 26):

$$\begin{bmatrix} 1 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 & 0 & 0 \\[1em] 0 & 0 & 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & \frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} D_{11} & C_{13} \\ \mathbf{0} & C_{23} \end{bmatrix} \tag{26}$$

where $\boldsymbol{D}_{12}$ is null. This matrix gives (eq. 27):

$$C_{23} = \begin{array}{c} \begin{array}{ccccc} \delta & \nu & A & \rho & \mu \end{array} \\ \begin{bmatrix} -1 & 2 & 1 & 1 & 0 \\ -1 & 1 & 1/2 & 0 & 1 \end{bmatrix} \end{array} \tag{27}$$

According to Theorem 5, the transpose C'_{23} of $\boldsymbol{C}_{23}$ is a complete B-matrix. Since there are five variables and since the rank of $\boldsymbol{D}$ is 3, Theorem 1 reveals that there are two dimensionless products in a complete set of B-numbers, each of which corresponds to a row of $\boldsymbol{C}_{23}$. This yields a functional relation between the two B-numbers as (eq. 28):

$$f\left(\frac{\nu^2 A\rho}{\delta}, \frac{\nu\mu A^{1/2}}{\delta}\right) = 0 \tag{28}$$

where $\pi_1 = \nu^2 A\rho/\delta$ and $\pi_2 = \nu\mu A^{1/2}/\delta$, or alternatively (eq. 29):

$$\frac{\nu^2 A\rho}{\delta} = f_1\left(\frac{\nu\mu A^{1/2}}{\delta}\right) \tag{29}$$

This relation is not in the best form for the calculation of the drag since δ appears in both products. Hence, it is necessary to change the two independent B-numbers by requiring that δ occur in only one of them. To this end, we let $\boldsymbol{M}$ be a nonsingular submatrix of C'_{23} of order 2 containing the row corresponding to δ. Thus, row 1 and, eg, row 5 of $\boldsymbol{C}_{23}$ are chosen to give

$$\boldsymbol{M} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \tag{30}$$

the adjoint matrix of which is given by (39) (eq. 31):

$$\boldsymbol{M}_a = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \tag{31}$$

Then, the matrix product (eq. 32):

$$\boldsymbol{C}'_{23}\boldsymbol{M}_a = \begin{bmatrix} -1 & -1 \\ 2 & 1 \\ 1 & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{array}{c} \delta \\ v \\ A \\ \rho \\ \mu \end{array} \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 1 & \frac{1}{2} \\ 1 & 1 \\ 0 & -1 \end{bmatrix} \tag{32}$$

is a desired B-matrix. The associated B-numbers are obtained as $\pi_1 = A\rho\nu^2/\delta$ and $\pi_2 = v\rho A^{1/2}/\mu$, yielding a functional relation (eq. 33):

$$\frac{\delta}{A\rho\nu^2} = f_2\left(\frac{\nu\rho A^{1/2}}{\mu}\right) \tag{33}$$

Let d be the diameter of the sphere. Then $A = \pi d^2/4$ and $\pi_2 = \pi^{1/2}\nu\rho d/2\mu$. The dimensionless product $\upsilon\rho d/\mu$, which was first derived by Osborne Reynolds, is the familiar Reynolds number, and is denoted by Re. Equation 33 can now be expressed as (eq. 34):

$$\delta = \left(\frac{A\rho\nu^2}{2}\right) f_3\,(\mathrm{Re}) \tag{34}$$

Defining a drag coefficient C_δ by an equation leads to equation 36:

$$\delta = C_\delta\left(\frac{A\rho\nu^2}{2}\right) \tag{35}$$

$$C_\delta = f_3(\mathrm{Re}) \tag{36}$$

Thus, the drag problem is reduced to an equation involving only two dimensionless products C_δ; and Re. The plot of the drag coefficient C_δ; as a function of the Re can be obtained from experimental data. By knowing the speed of the sphere, equation 34 together with the drag coefficient is now in the best form for the direct determination of the drag (see also RHEOLOGICAL MEASUREMENTS).

On the other hand, suppose that the speed is to be determined when the drag is given. Then equations 33 or 34 are not convenient, and the two independent B-numbers must be changed again, so that the speed ν will occur only in one of the B-numbers. To this end, let $\boldsymbol{M}$ be a nonsingular submatrix of C'_{23} of order 2 containing the row corresponding to ν. Thus, choosing row 2 and, again say, row 5 of the transpose of the matrix of equation 27 gives (eq. 37):

$$\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \tag{37}$$

Then the matrix product (eq. 38):

$$-C'_{23}\boldsymbol{M}_a = -\begin{bmatrix} -1 & -1 \\ 2 & 1 \\ 1 & \frac{1}{2} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{array}{c} \delta \\ v \\ A \\ \rho \\ \mu \end{array} \begin{bmatrix} 1 & 1 \\ -2 & 0 \\ -1 & 0 \\ -1 & 1 \\ 0 & -2 \end{bmatrix} \tag{38}$$

is a desired B-matrix, where $\boldsymbol{M}_a$ is the adjoint matrix of $\boldsymbol{M}$. The associated B-numbers become $\pi_1 = \delta/A\rho\nu^2$ and $\pi_2 = \delta\rho/\mu^2$ yielding (eq. 39):

$$\frac{\delta}{A\rho\nu^2} = f_4\left(\frac{\delta\rho}{\mu^2}\right) \tag{39}$$

From equation 35, it is simple to demonstrate that (eq. 40):

$$\frac{\nu\rho d}{\mu} = \mathrm{Re} = \frac{\sqrt{\frac{8\rho\delta}{C_\delta\pi}}}{\mu} \tag{40}$$

and that equation 39 can be expressed as (eq. 41):

$$C_\delta = f_5\left(\frac{\rho\delta}{\mu^2}\right) \tag{41}$$

The drag coefficient C_δ; can be plotted as a function of the dimensionless product $\rho\,\delta/\mu^2$. Thus, equations 40 and 41 are in proper form for direct determination of the speed once the drag is given.

Suppose that an experiment was set up to determine the values of drag for various combinations of ν, A, ρ, and μ. If each variable is to be tested at 10 values, then it would require $10^4 = 10,000$ tests for all combinations of these values. On the other hand, as a result of dimensional analysis the drag can be calculated by means of the drag coefficient, which, being a function of Re, can be uniquely determined by the values of Re. Thus, for data of equal accuracy, it now requires only 10 tests at ten different values of Re instead of 10,000, a remarkable saving in experiments.

In addition, dimensional analysis can be used in the design of scale experiments. For example, if a spherical storage tank of diameter d is to be constructed, the problem is to determine wind load at a velocity v. Equations 34 and 36 indicate that, once the drag coefficient C_δ is known, the drag can be calculated from C_δ immediately. But C_δ is uniquely determined by the value of Re. Thus, a scale model can be set up to simulate the Re number of the spherical tank. To this end, let a sphere of diameter $\hat{d}$ be immersed in a fluid of density $\hat{\rho}$ and viscosity $\hat{\mu}$ and towed at the speed of $\hat{\nu}$. Requiring that this model experiment have the same Re number as the spherical storage tank gives

$$\frac{\hat{\nu}\hat{\rho}\hat{d}}{\hat{\mu}} = \frac{\nu\rho d}{\mu} \tag{42}$$

where ρ and μ are the air density and viscosity, respectively. Thus, a sphere of 1 m in diameter, immersed in water and towed at 32.5 km/h has the same Re number as the spherical storage tank of 10 m in diameter with air flowing over it at 50 km/h. By towing a smaller sphere in a water tank and measuring its drag force $\hat{\delta}$, the drag coefficient is determined from equation 35 by the formula

$$C_\delta = \frac{8\hat{\delta}}{\pi\hat{\rho}\hat{d}^2\hat{\nu}^2} \tag{43}$$

This C_δ can then be used in the original full-scale spherical storage tank to calculate its wind load (eq. 35):

$$\delta = \frac{\pi\rho d^2\nu^2 C_\delta}{8} \tag{44}$$

Example 3. The mean free path P_m of electrons scattered by a crystal lattice is known to involve temperature θ, energy E, the elastic constant C, the Planck's constant h, the Boltzmann constant k, and the electron mass M [see, eg, Ref. (25)]. The problem is to derive a general equation among these variables.

Again length l, mass m, time t, and temperature T are chosen as the reference dimensions. Then the associated dimensional matrix $\boldsymbol{D}$ is obtained as (eq. 45):

$$\boldsymbol{D} = \begin{array}{c} \\ l \\ m \\ t \\ T \end{array} \begin{array}{c} \begin{array}{ccccccc} P_m & C & E & h & k & \theta & M \end{array} \\ \begin{bmatrix} 1 & -1 & 2 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -2 & -2 & -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \end{bmatrix} \end{array} \tag{45}$$

To compute a complete B-matrix, the augmented matrix (eq. 46):

$$[\boldsymbol{D}' \quad \boldsymbol{I}_7] \tag{46}$$

is considered, which is given by (eq. 47):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & -2 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{47}$$

and which may be put in the form of equation 22 by a sequence of elementary row operations yielding (eq. 48):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & -5 & -1 & 0 & 2 & 0 & 0 & -1 \end{bmatrix} \tag{48}$$

from which the appropriate submatrices can be identified: $\boldsymbol{D}_{12}$ is null and (eqs. 49–51):

$$\boldsymbol{D}_{11} = \begin{array}{c} (\text{row } 1) \\ (\text{row } 2) \\ (\text{row } 4) \\ (\text{row } 5) \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{49}$$

$$\boldsymbol{C}_{13} = \begin{array}{c} (\text{row } 1) \\ (\text{row } 2) \\ (\text{row } 4) \\ (\text{row } 5) \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & -1 \\ 2 & 0 & 0 & -2 & 1 & 0 & 1 \end{bmatrix} \tag{50}$$

$$\boldsymbol{C}_{23} = \begin{array}{c} \\ (\text{row } 3) \\ (\text{row } 6) \\ (\text{row } 7) \end{array} \begin{array}{c} \begin{array}{ccccccc} P_m & C & E & h & k & \theta & M \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & -2 & 0 & 0 & 1 \\ 2 & 0 & 0 & -2 & 1 & 1 & 1 \\ -5 & -1 & 0 & 2 & 0 & 0 & -1 \end{bmatrix} \end{array} \tag{51}$$

Thus, the transpose of $\boldsymbol{C}_{23}$ is a complete B-matrix. Since there are seven variables involved in the phenomenon and the rank of $\boldsymbol{D}$ is 4, from Theorem 1 there are three dimensionless products in a complete set of B-numbers, each of which corresponds to a row of $\boldsymbol{C}_{23}$.

Suppose that the problem is to find a B-matrix of $\boldsymbol{D}$ such that the variables P_m, C, and E each occur in one and only one of the B-vectors. Since the submatrix $\boldsymbol{M}$ of C'_{23} consisting of the first three rows corresponding to the variables P_m, C, and E is nonsingular, according to Theorem 6 there exists a B-matrix with the desired property. Let $\boldsymbol{M}_a$ be the adjoint matrix of $\boldsymbol{M}$. Then (eq. 52):

$$\boldsymbol{C}'_{23}\boldsymbol{M}_a = \begin{bmatrix} 2 & 2 & -5 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ -2 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ -1 & 5 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \\ 2 & -6 & 0 \\ -1 & 5 & 2 \\ -1 & 5 & 2 \\ -1 & 3 & 0 \end{bmatrix} \tag{52}$$

Hence, the right-hand side of equation 52 is a desired complete B-matrix. A functional relationship among the associated B-numbers can be obtained and is given by (eq. 53):

$$f\left(\frac{h^2}{P_m^2 k\theta M}, \frac{k^5\theta^5 M^3}{C^2 h^6}, \frac{k^2\theta^2}{E^2}\right) = 0 \tag{53}$$

Observe that the variables P_m, C, and E each occur in only one dimensionless product. Alternatively, equation 53 can be written as (eq. 54):

$$P_m^2 = \left(\frac{h^2}{k\theta M}\right) f_1\left(\frac{k^5\theta^5 M^3}{C^2 h^6}, \frac{k^2\theta^2}{E^2}\right) \tag{54}$$

The functional relation in equation 53 or 54 cannot be determined by dimensional analysis alone; it must be supplied by experiments. The significance is that the mean-free-path problem is reduced from an original relation involving seven variables to an equation involving only three dimensionless products, a considerable saving in terms of the number of experiments required in determining the governing equation.

5. Optimization of the Complete B-Matrices

In the foregoing, the computation of a complete B-matrix from a given dimensional matrix has been indicated. With the exception that certain variables may each be required to occur in only one dimensionless product, the selection of a complete B-matrix is totally arbitrary. In order to simplify the formulas associated with a complete B-matrix and to provide a procedure for establishing an explicit set of B-numbers, it is necessary to impose additional constraints in the selection of these B-vectors in forming a complete B-matrix. To avoid the fractional exponents of the formulas, the elements of the matrices are restricted to integers. In addition, the following criteria are proposed for the optimization of the B-matrices: (1) maximize the number of zeros in a complete B-matrix, and (2) minimize the sum of the absolute values of all the integers of a complete B-matrix. These criteria (21) are chosen so that the formulas associated with a physical phenomenon are in their simplest form, otherwise they are completely arbitrary. Evidently the order of the two-optimization criteria is important. For the purpose of this article the sequence consisting of criterion 1 followed by criterion 2 is assumed.

Let $\boldsymbol{A}$ be a complete B-matrix of order $n \times m$ and $\boldsymbol{A}_{qm}$ be a submatrix of order $q \times m$ and of rank $m-1$ in $\boldsymbol{A}$. Denote by $S(\boldsymbol{A}_{qm})$ the set of all submatrices of the form $\boldsymbol{A}_{qm} \in \boldsymbol{A}$. Write

$$S(\boldsymbol{A}_{q_i m}) > S(\boldsymbol{A}_{q_j m}) \tag{55}$$

if and only if $q_i > q_j$, and $q_i > q_j$ if and only if $i < j$. For each $\boldsymbol{A}_{q_i m} \in S(\boldsymbol{A}_{q_i m})$, the associated system of homogeneous linear equations

$$\boldsymbol{A}_{q_i m}\boldsymbol{X}_i = \mathbf{0} \tag{56}$$

has nullity one, where $\boldsymbol{X}_i$ is an m-row column vector or simply an m vector. Thus, if $\boldsymbol{C}_{qi}$ is an integer solution of equation 56, then $h\boldsymbol{C}_{qi}$ is its complete integer solution for some real scalar h. In order to provide a procedure for establishing an explicit and unique solution, choose $1/h$ to be the largest common factor (integer) among all the integers contained in the vector $\bar{\mathbf{A}}_{q_i m}\mathbf{C}_{q_i}$, where $\bar{\mathbf{A}}_{q_i m}$ is the matrix obtained from $\boldsymbol{A}$ by deleting all the rows contained in $\bar{\boldsymbol{A}}_{q_i m}$. The unique vector $h\boldsymbol{C}_{q_i}$ is called the associated vector of $\boldsymbol{A}_{q_i m}$ and the matrix product $\boldsymbol{A}(h\boldsymbol{C}_{q_i})$ the associated B-vector of $\boldsymbol{A}_{q_i m}$. For $\boldsymbol{A}^u_{q_i m}$, $\boldsymbol{A}^v_{q_i m} \in S(\boldsymbol{A}_{q_i m})$, let $\boldsymbol{B}^u_{q_i}$ and $\boldsymbol{B}^v_{q_i}$ be the associated B-vectors of $\boldsymbol{A}^u_{q_i m}$ and $\boldsymbol{A}^u_{q_i m}$, respectively. Define

$$\boldsymbol{A}^u_{q_i m} > \boldsymbol{A}^v_{q_i m} \tag{57}$$

if $u < v$ and the sum of the absolute values of all the entries of $\boldsymbol{B}^u_{q_i}$ is not greater than that of $\boldsymbol{B}^v_{q_i}$. Obviously, this ordering is not unique if $\boldsymbol{B}^u_{q_i}$ and $\boldsymbol{B}^v_{q_i}$ have the same sum.

As an illustration, consider the B-matrix of equation 52, the submatrices $\boldsymbol{A}_{43}$ of which is found to be

$$\boldsymbol{A}^1_{43} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 2 & -6 & 0 \\ -1 & 3 & 0 \end{bmatrix} \qquad \boldsymbol{A}^2_{43} = \begin{bmatrix} 2 & -6 & 0 \\ -1 & 5 & 2 \\ -1 & 5 & 2 \\ -1 & 3 & 0 \end{bmatrix} \tag{58}$$

Their associated vectors $\boldsymbol{C}^1_4$ and $\boldsymbol{C}^2_4$ of the associated system of homogeneous linear equations

$$\boldsymbol{A}^1_{43}X_1 = 0 \qquad \boldsymbol{A}^2_{43}X_2 = 0 \tag{59}$$

are computed as

$$h_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad h_2\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \tag{60}$$

where the real scalars h_1 and h_2 are found to be $\frac{1}{2}$, giving

$$\boldsymbol{C}_4^1 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix} \qquad \boldsymbol{C}_4^2 \begin{bmatrix} -3/2 \\ -1/2 \\ 1/2 \end{bmatrix} \tag{61}$$

The associated B-vectors of equation 58 are obtained as

$$\boldsymbol{B}_4^1 = [0, \quad 0 \quad -1 \quad 0 \quad 1 \quad 1 \quad 0]' \qquad \boldsymbol{B}_4^2 = [-3 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]' \tag{62}$$

where the prime denotes the matrix transpose. Since the sum of the absolute values of all the entries in $\boldsymbol{B}_4^1$ is less than that of $\boldsymbol{B}_4^2$, the elements of $S(\boldsymbol{A}_{43})$ are ordered as

$$S(\boldsymbol{A}_{43}) = \{\boldsymbol{A}_{43}^1, \quad \boldsymbol{A}_{43}^2\} \tag{63}$$

5.1. Theorem 7. The associated B-vectors of a set of submatrices of a complete B-matrix are linearly independent if and only if the associated vectors of these submatrices are linearly independent.

Since the columns of any complete B-matrix are a basis for the null space of the dimensional matrix, it follows that any two complete B-matrices are related by a nonsingular transformation. In other words, a complete B-matrix itself contains enough information as to which linear combinations should be formed to obtain the optimized ones. Based on this observation, an efficient algorithm for the generation of an optimized complete B-matrix is available (22). No attempt is made here to justify the algorithm. Instead, an optimization algorithm is described and an example is being used to illustrate the results.

6. Optimization Algorithm

The algorithm can be used to generate all complete optimized B-matrices. The steps of the procedure are given as follows:

Step 1. For a given dimensional matrix $\boldsymbol{D}$, compute a complete B-matrix $\boldsymbol{A}$ of order $n \times m$.

Step 2. Generate the sets $S^*(\boldsymbol{A}_{q_1 m})$, $S^*(\boldsymbol{A}_{q_2 m}), \ldots$ from $\boldsymbol{A}$, and order them properly with $q_1 > q_2 > \ldots$, where

$$S^*(\boldsymbol{A}_{q_j m}) = \left\{ \boldsymbol{A}_{q_j m} \middle| \begin{array}{l} \boldsymbol{A}_{q_j m} \in S(\boldsymbol{A}_{q_j m}) \text{ and there is no } \boldsymbol{A}_{q_i m} \in S(\boldsymbol{A}_{q_i m}), \\ q_i > q_j, \text{ suth that } \boldsymbol{A}_{q_j m} \text{ is a submatrix of } \boldsymbol{A}_{q_i m} \end{array} \right\}, \quad j = 1, 2... \tag{64}$$

In most cases, the first one or two such sets $S^*(\boldsymbol{A}_{q_1 m})$ and $S^*(\boldsymbol{A}_{q_2 m})$ are sufficient for the purposes.

Step 3. Compute the associated vectors of the elements of $S^*(\boldsymbol{A}_{q_1m})$, $S^*(\boldsymbol{A}_{q_2m}), \ldots$

Step 4. Compute the associated B-vectors of the elements of $S^*(\boldsymbol{A}_{q_1m})$, $S^*(\boldsymbol{A}_{q_2m}), \ldots$

Step 5. Order the elements of $S^*(\boldsymbol{A}_{qim})$ properly, ie, if $\boldsymbol{A}^u_{q_im}, \boldsymbol{A}^v_{q_im} \in S^*(\boldsymbol{A}_{qim})$ and if $u < v$, then the sum of the absolute values of all the entries of $\boldsymbol{B}^u_{q_i}$ is not greater than that of $\boldsymbol{B}^v_{q_i}$, where $\boldsymbol{B}^u_{q_i}$ and $\boldsymbol{B}^v_{q_i}$ are the associated B-vectors of $\boldsymbol{A}^u_{q_im}$ and $\boldsymbol{A}^v_{q_im}$, respectively.

Step 6. Arrange the associated B-vectors of the elements of the sets $S^*(\boldsymbol{A}_{qim})$ in the following matrix form

$$[\boldsymbol{B}^1_{q_1} \quad \boldsymbol{B}^2_{q_1} \cdots \boldsymbol{B}^1_{q_2} \quad \boldsymbol{B}^2_{q_2} \cdots \boldsymbol{B}^1_{q_3} \quad \boldsymbol{B}^2_{q_3} \quad \cdots \quad \cdots] \tag{65}$$

Step 7. Select a submatrix of order $n \times m$ and of rank m from equation 65 consisting of as many left-hand side columns as possible. This matrix is a complete optimized B-matrix.

Example 4. For a given lattice, a relationship is to be found between the lattice resistivity and temperature using the following variables: mean free path L, the mass of electron M, particle density N, charge Q, Planck's constant h, Boltzmann constant k, temperature θ, velocity ν, and resistivity ρ. Suppose that length l, mass m, time t, charge q, and temperature T are chosen as the reference dimensions. The dimensional matrix $\boldsymbol{D}$ of the variables is given by (eq. 66):

$$\boldsymbol{D} = \begin{array}{c} \\ l \\ m \\ t \\ q \\ T \end{array} \begin{array}{c} \begin{array}{ccccccccc} L & M & N & Q & h & k & \theta & \nu & \rho \end{array} \\ \begin{bmatrix} 1 & 0 & -3 & 0 & 2 & 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \end{bmatrix} \end{array} \tag{66}$$

Step 1. By using the procedure outlined in the preceding section, a matrix similar to that of equation 22 is obtained as follows (eq. 67):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -2 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & -2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{67}$$

Thus, a complete B-matrix $\boldsymbol{A}$ of $\boldsymbol{D}$ is obtain as

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{68}$$

Step 2. Generate the set $S^*(\boldsymbol{A}_{74})$ and $S^*(\boldsymbol{A}_{54})$ from $\boldsymbol{A}$.

$$S^*(\boldsymbol{A}_{74}) = \{\boldsymbol{A}_{74}\} \tag{69}$$

$$S^*(\boldsymbol{A}_{54}) = \{\boldsymbol{A}_{54}^1, \boldsymbol{A}_{54}^2, \boldsymbol{A}_{54}^3, \boldsymbol{A}_{54}^4, \boldsymbol{A}_{54}^5, \boldsymbol{A}_{54}^6\} \tag{70}$$

where

$$\boldsymbol{A}_{54}^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{A}_{54}^2 = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{A}_{54}^3 = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_{54}^4 = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{A}_{54}^5 = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_{54}^6 = \begin{bmatrix} 3 & 2 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \boldsymbol{A}_{74} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & -2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3. The associated vectors $\boldsymbol{C}_7$ of $\boldsymbol{A}_{74}$ and $\boldsymbol{C}_5^i$ of $\boldsymbol{A}_{54}^i$, $i = 1, 2, 3, 4, 5, 6$ are found to be

$$\boldsymbol{C}_7 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{C}_5^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{C}_5^2 = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \boldsymbol{C}_5^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{71a}$$

$$\boldsymbol{C}_5^4 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix} \boldsymbol{C}_5^5 = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \end{bmatrix} \boldsymbol{C}_6^5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \tag{71b}$$

Step 4. The associated B-vectors $\boldsymbol{B}_7$ and $\boldsymbol{C}_5^i$ of $\boldsymbol{A}_{74}$ and $\boldsymbol{A}_{54}^i$, respectively, are computed as follows:

$$\boldsymbol{B}_7 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{B}_7^1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \boldsymbol{B}_7^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \boldsymbol{B}_7^3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \boldsymbol{B}_7^4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 0 \end{bmatrix} \boldsymbol{B}_7^5 = \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 3 \\ 0 \\ 0 \\ -3 \\ 0 \end{bmatrix} \boldsymbol{B}_7^6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 6 \\ -3 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix} \tag{72}$$

Step 5. Since the sums of the absolute values of all the entries of $\boldsymbol{B}_5^i$, $i = 1$, 2, 3, 4, 5, 6 are given by 4, 5, 5, 5, 10, 13, respectively, the order of the submatrices $\boldsymbol{A}_{54}^2$, $\boldsymbol{A}_{54}^3$, and $\boldsymbol{A}_{54}^4$ could be arranged in any order. For convenience, let

$$\boldsymbol{A}_{54}^1 > \boldsymbol{A}_{54}^2 > \boldsymbol{A}_{54}^3 > \boldsymbol{A}_{54}^4 > \boldsymbol{A}_{54}^5 > \boldsymbol{A}_{54}^6 \tag{73}$$

Step 6. Arrange the associated B-vectors of the elements of the sets $S^*(\boldsymbol{A}_{74})$ and $S^*(\boldsymbol{A}_{54})$ in the following matrix form

$$[\boldsymbol{B}_7 \quad \boldsymbol{B}_5^1 \quad \boldsymbol{B}_5^2 \quad \boldsymbol{B}_5^3 \quad \boldsymbol{B}_5^4 \quad \boldsymbol{B}_5^5 \quad \boldsymbol{B}_5^6] \tag{74}$$

Step 7. Since $\boldsymbol{B}_7$, $\boldsymbol{B}_5^1$, $\boldsymbol{B}_5^2$ and $\boldsymbol{B}_5^3$ $\boldsymbol{B}_5^3$ are linearly independent, it follows that the matrix formed by these columns is an optimized B-matrix of $\boldsymbol{D}$, the transpose of which is given by (eq. 75):

$$\begin{array}{c} \begin{array}{ccccccccc} L & M & N & Q & h & k & \theta & \nu & \rho \end{array} \\ \begin{bmatrix} 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{75}$$

Thus, the general relationship between the lattice resistivity and temperature can be expressed as (eq. 76):

$$f\left(L^3N, \frac{LMv}{h}, \frac{Mv^2}{k\theta}, \frac{Q^2\rho}{Lh}\right) = 0 \tag{76}$$

The number of independent variables is reduced from the original nine to four, which is a great saving in terms of the number of experiments required to determine the desired function. For example, suppose that a decision is made to test only four values for each variable. Then it would require 4^9=262,144 experiments to test all combinations of these values in the original equation. As a result of equation 76, only 4^4=256 tests are now required for four values each of the four B-numbers.

7. Conclusion

Dimensional analysis provides a means of judicious grouping of variables associated with a physical phenomenon to form dimensionless products of these variables without destroying the generality of the relationship. Consequently, the equation describing the physical phenomenon may be more easily determined experimentally. In addition, it guides the experimenter in the selection of experiments capable of yielding significant information and in the avoidance of redundant experiments, and makes possible the use of scale models for experiments. The technique is particularly useful when the problems involve a large number of variables.

8. Nomenclature

A	area or acceleration
C	elastic constant
C_δ	drag coefficient
$\hat{d}, d$	diameter
D, $\hat{D}$, $\tilde{D}$	dimensional matrix of variables
E	energy
f, F	force
g	acceleration of gravity
h	Planck's constant
k	Boltzmann constant
L	length or mean free path
l	length
m, M	mass
N	particle density
P	period
P_m	mean free path
q, Q	charge
Re	Reynolds number
θ, T	temperature
t	time
$\hat{v}$, v	velocity or speed
δ	drag
ϵ	permittivity
$\hat{\mu}$, μ	permeability or viscosity
$\hat{\rho}$, ρ	fluid density or resistivity

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